Definitions and key facts for section 4.1

Definition: A **vector space** is a nonempty set V of objects, called *vectors*, on which two operations are defined: *addition* and *multiplication by scalars* (real numbers) which obey the following ten axioms. The axioms hold of all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero vector $\mathbf{0}$** in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces.

1. The space $\mathbb{R}^n$ for $n \geqslant 1$ is the fundamental example of a vector space. We rely on the geometric intuition developed here to understand and visualize many concepts of general vector spaces.
2. Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule (from section 1.3), and for each v in V, define cv to be the arrow whose length is $|c|$ times the length of v, pointing in the same direction as v if $c \geqslant 0$ and otherwise pointing in the opposite direction. Notice V is a vector space.
3. Let $M_{m\times n}$ be the set of all $m \times n$ matrices with entires in $\mathbb{R}$. With addition and scalar multiplication defined as in section 2.1 $M_{m\times n}$ is a vector space.
4. Let $\mathbb{P}_n$ be the set of all polynomials of degree *at most* n. Addition is defined by combining like coefficients and scalar multiplication by scaling each coefficient. Note $\mathbb{P}_n$ is a vector space.
5. Let W be the set of all real-valued functions on $\mathbb{R}$. For two such functions f and g, define addition by $(f + g)(x) = f(x) + g(x)$ and scalar multiplication by $(cf)(x) = c(f(x))$. Then W is a vector space.
6. Let $\mathbb{S}$ be the set of all doubly infinite sequences of numbers
$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots).$$
Define addition of two such sequences $\{y_k\} + \{z_k\}$ as $\{y_k + z_k\}$. Define scalar multiplication as $c\{y_k\} = \{cy_k\}$. Then $\mathbb{S}$ is a vector space.

In class, we verify a vector space axiom or two for each of the above six vector spaces. OF course, all six satisfy all ten axioms.

Definition: A **subspace** of a vector space V is a subset H of V that has three properties

1. The zero vector of V is in H.
2. H is closed under vector addition. (That is, if $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v}$ is also in H.)
3. H is closed under multiplication by scalars. (That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.)

Examples of subspaces.

1. The set $\{\mathbf{0}\}$ is a subspace of any vector space, called the **zero subspace**.
2. Let $H = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$. Then H is a subspace of $M_{2\times 2}$.
3. The set $H = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$ is a subspace of W.
4. Notice that $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$. However, $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.
5. The disc $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$ is *not* a subspace of $\mathbb{R}^2$.
6. For any $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, the set $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of V.

Notice the previous example can be extended to show the following.

Fact: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

We call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the **subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a **spanning** (or **generating**) **set** for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

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7. Let $H = \{(a - 3b, b - a, a, b) : a, b \text{ in } \mathbb{R}\}$. Then H is a subspace of $\mathbb{R}^4$.
8. Show $H = \{at^2 + at + a : a \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{P}_n$ for $n \geqslant 2$.

In class we verify each example.